

On Signature Transition and Compactification in Kaluza-Klein Cosmology

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Abstract

We consider an empty (4+1) dimensional Kaluza-Klein universe with a negative cosmological constant and a Robertson-Walker type metric. It is shown that the solutions to Einstein field equations have degenerate metric and exhibit transitions from a Euclidean to a Lorentzian domain. We then suggest a mechanism, based on signature transition which leads to compactification of the internal space in the Lorentzian region as $a \sim |\Lambda|^{1/2}$. With the assumption of a very small value for the cosmological constant we find that the size of the universe R and the internal scale factor a would be related according to $Ra \sim 1$ in the Lorentzian region. The corresponding Wheeler-DeWitt equation has exact solution in the mini-superspace giving rise to a quantum state which peaks in the vicinity of the classical solutions undergoing signature transition.

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1 Introduction

The question of signature transition in classical and quantum cosmological models has been of some interest in the past few years. It was first addressed in the work of Hartle and Hawking [1] in which they examined quantum cosmologies admitting quantum amplitudes in the form of a sum over all compact Riemannian manifolds whose boundaries coincide with the loci of signature change. Other workers have also investigated this question and considered signature transition in general relativity by adopting model theories which mostly rely on Einstein's field equations coupling to a scalar field [2]. The solution of the resulting field equations under a properly parametrized metric, when interpreted suitably, would then indicate a change of signature.

A particular model, relevant to the present work, is that of Dereli and Tucker [4] in which a self interacting scalar field is coupled to Einstein's field equations with a potential containing a Sinh-Gordon scalar interaction. These equations are then solved exactly for the scalar field and the scale factor as dynamical variables, giving rise to cosmological solutions with a degenerate metric, describing a continuous signature transition from a Euclidean domain to a Lorentzian space-time in a spatially flat Robertson-Walker cosmology. The corresponding quantum cosmology has also been investigated [5] where the Wheeler-DeWitt equation arises from an anisotropic oscillator-ghost-oscillator vanishing Hamiltonian. It is solved exactly and leads to normalizable states with the quantum states constructed as belonging to distinct Hilbert subspaces each of which being characterized by a particular "quantization" condition on the parameters of the scalar field potential. These quantum states correspond to excited quantum cosmologies and relate to classical solutions without resort to WKB approximation techniques. A similar analysis of the classical and quantum theory of a scalar (dilaton) field interacting with gravity has been reported in two dimensions [6] in which a class of analytic solutions to the Wheeler-DeWitt equation relate, in a remarkable way, to the general solution of the classical field equations.

In this paper, we consider a (4+1) dimensional Kaluza-Klein cosmology with a *negative* cosmological constant and a Robertson-Walker type metric having two dynamical variables, the usual scale factor R and the internal scale factor a . Following [4] and [5], we insist on a preferred coordinate that controls the evolution of signature dynamics, seeking suitably smooth continuous solutions for R and a passing through the hypersurface of signature change. These classical solutions admitting signature transition would then suggest a compactification mechanism for the internal scale factor a . Some authors [8, 9, 10] have already considered various compactification mechanisms in different models. In particular in [9], the author has suggested a compactification mechanism based on signature change for a *positive* cosmological constant. Here, we will discuss the differences and similarities between these models and the one presented here in order to see how the results can be compared. We then find exact solutions to the corresponding Wheeler-DeWitt equation arising from an isotropic oscillator-ghost-oscillator vanishing Hamiltonian. Due to this isotropy, there is no "quantization" condition and thus no excited cosmologies. However, we show that the desired quantum state, identified with a non-dispersive wave packet, relates to classical solutions in that it peaks in the vicinity of the classical loci corresponding to this cosmology.

2 Classical Cosmology

We start with the metric considered in [3] in which the space-time is assumed to be of Robertson-Walker type having a compactified space which is assumed to be the circle S^1 . In this paper we adopt the real chart $\{\beta, r^1, r^2, r^3, \rho\}$ with β , r^i and ρ denoting the lapse function, the space coordinates and the compactified space coordinate respectively. We therefore take

$$ds^2 = -\beta d\beta^2 + \bar{R}^2(\beta) \frac{dr^i dr^i}{(1 + \frac{kr^2}{4})^2} + \bar{a}^2(\beta) d\rho^2, \quad i = 1, 2, 3 \quad (1)$$

where $k = 0, \pm 1$ and $\bar{R}(\beta)$ is the scale factor and $\bar{a}(\beta)$ is the radius of the compactified space, both of which are assumed to depend only on the lapse function β . The signature of the metric is Lorentzian for $\beta > 0$ and Euclidean for $\beta < 0$. For positive values of β (Lorentzian region), one can recover the cosmic time by writing $t = \frac{2}{3}\beta^{\frac{3}{2}}$, leading to

$$ds^2 = -dt^2 + R^2(t) \frac{dr^i dr^i}{(1 + \frac{kr^2}{4})^2} + a^2(t) d\rho^2 \quad (2)$$

where $R(t) = \bar{R}(\beta(t))$ and $a(t) = \bar{a}(\beta(t))$ in the $\{t, r^i, \rho\}$ chart. As in [4], we formulate our differential equations in a region that does not include $\beta = 0$ and seek real solutions for R and a smoothly passing through the $\beta = 0$ hypersurface. The curvature scalar corresponding to metric (1) is obtained as

$$\mathcal{R} = 6 \left[\frac{\ddot{R}}{R} + \frac{k + \dot{R}^2}{R^2} \right] + 2 \frac{\ddot{a}}{a} + 6 \frac{\dot{R} \dot{a}}{R a} \quad (3)$$

where a dot represents differentiation with respect to t . Substituting this result into Einstein-Hilbert action with a cosmological constant Λ

$$I = \int \sqrt{-g} (\mathcal{R} - \Lambda) dt d^3r d\rho \quad (4)$$

and integrating over spatial dimensions gives an effective Lagrangian L in the mini-superspace (R, a) as

$$L = \frac{1}{2} R a \dot{R}^2 + \frac{1}{2} R^2 \dot{R} \dot{a} - \frac{1}{2} k R a + \frac{1}{6} \Lambda R^3 a. \quad (5)$$

3 Solutions

By defining $\omega^2 \equiv -\frac{2\Lambda}{3}$ and changing the variables as [7]

$$u = \frac{1}{\sqrt{8}} \left[R^2 + R a - \frac{3k}{\Lambda} \right], \quad v = \frac{1}{\sqrt{8}} \left[R^2 - R a - \frac{3k}{\Lambda} \right] \quad (6)$$

L takes on the form

$$L = \frac{1}{2} \left[(\dot{u}^2 - \omega^2 u^2) - (\dot{v}^2 - \omega^2 v^2) \right]. \quad (7)$$

A point (u, v) in this mini-superspace represents a 4-geometry. The classical equations of motion are given by

$$\ddot{u} = -\omega^2 u, \quad \ddot{v} = -\omega^2 v. \quad (8)$$

Choosing the initial conditions $\dot{u}(0) = \dot{v}(0) = 0$, the solutions are obtained as

$$u(t) = A \cosh \left(\sqrt{\frac{2\Lambda}{3}} t \right), \quad v(t) = B \cosh \left(\sqrt{\frac{2\Lambda}{3}} t \right) \quad (9)$$

where A and B are constants to be determined later. Assuming the full (4+1) dimensional Einstein equations to hold, this implies that the Hamiltonian corresponding to L in (7) must vanish, that is

$$H = \frac{1}{2} \left[(\dot{u}^2 + \omega^2 u^2) - (\dot{v}^2 + \omega^2 v^2) \right] = 0 \quad (10)$$

which describes an isotropic oscillator-ghost-oscillator system. Now, the solutions (9) must satisfy the constraint of vanishing Hamiltonian. Thus, substitution of equations (9) into (10) gives a relation between the constants A and B

$$A = \pm B \quad (11)$$

implying that we can rewrite the solutions (9) as

$$u(t) = A \cosh \left(\sqrt{\frac{2\Lambda}{3}} t \right), \quad v(t) = \epsilon A \cosh \left(\sqrt{\frac{2\Lambda}{3}} t \right) \quad (12)$$

where ϵ takes the values ± 1 according to the choices in (11). Classical solutions (12) may be displayed as trajectories $u = \pm v$ in the mini-superspace (u, v) . We recover $R(t)$ and $a(t)$ from $u(t)$ and $v(t)$ as

$$\begin{aligned} R(t) &= \left[\sqrt{2}(u(t) + v(t)) + \frac{3k}{\Lambda} \right]^{1/2} \\ a(t) &= \left[\sqrt{2}(u(t) + v(t)) + \frac{3k}{\Lambda} \right]^{-1/2} \left[\sqrt{2}(u(t) - v(t)) \right]. \end{aligned} \quad (13)$$

For $\epsilon = -1$ one finds the solutions in terms of β as

$$\begin{aligned} R_c &= \sqrt{\frac{3k}{\Lambda}} \\ a(\beta) &= \sqrt{\frac{\Lambda}{3k}} \cosh \left(\sqrt{\frac{2\Lambda}{3}} \frac{2}{3} \beta^{3/2} \right) \end{aligned} \quad (14)$$

where $A = \frac{1}{\sqrt{8}}$ is taken for convenience¹ and solutions (12) have been used. Also for $\epsilon = +1$, one finds the solutions

$$\begin{aligned} R(\beta) &= \left[\cosh \left(\sqrt{\frac{2\Lambda}{3}} \frac{2}{3} \beta^{3/2} \right) + \frac{3k}{\Lambda} \right]^{1/2} \\ a &= 0. \end{aligned} \quad (15)$$

¹ Note that the dimension of A is (length)².

4 Signature Transition and Compactification

In this section we discuss signature transition occurring in the model presented above and show that it induces compactification on the internal space. However, before doing so we give a brief history of the mechanisms of compactification related to the present work, in particular the recent works discussed in [8] and [9].

In [9], it is shown that for a higher dimensional FRW model with $S^3 \times S^6$ as spatial sections with two scale factors a_1, a_2 and a positive cosmological constant, the classical signature change induces compactification. This is done by a dynamical mechanism that drags the size of S^6 down and gives rise to a long-time stability at an unobservably small scale. This mechanism is based on the existence of a signature transition and the interplay between the causal structure of the Wheeler-DeWitt metric and the sign of the corresponding potential W appearing in the action defined in the mini-superspace (a_1, a_2) . In order to diagonalize the kinetic term in the action a new set of variables (u, v) are defined. In this new mini-superspace the curve satisfying $W = 0$ consists of two branches which are connected smoothly. In one branch, the compactification occurs for S^6 and in the other, it occurs for S^3 . For the compactification of S^6 , one finds $a_1(t)$ oscillating around some linearly growing average, whereas $a_2(t)$ performs damped oscillations around a limiting value of order $\Lambda^{-\frac{1}{2}}$, both solutions being stable against small perturbations. The effective five-dimensional space-time metric, obtained by taking the proper time average, has a Lorentzian signature, undergoes exponential inflation (in Λ) in S^3 and induces compactification on S^6 of order $\Lambda^{-\frac{1}{2}}$. This requires a large cosmological constant in order to be consistent with the unobservability of the compactified dimensions. On the other hand, stopping inflation requires switching off Λ , but then the radius of the compactified S^6 will blow up. Usually, this type of problem is expected in the presence of a large positive cosmological constant whose possible solutions are suggested in [9].

In [8] however, the compactification mechanism is studied for a $D+1$ dimensional toroidally compact Kaluza-Klein cosmology with a negative cosmological constant consisting of matter that is either dust or coherent excitations of a dilaton field. In this model, compactification is done by diagonalizing the classical Hamiltonian which leads to a new D -dimensional mini-superspace. Applying the Heisenberg equations of motion and taking the expectation values, one finds the cosmic time dependence of the expectation values of the mini-superspace variables. It is then shown that the expectation values of some of the dimensions show a quantum inflationary phase while simultaneously the remaining dimensions show a quantum deflationary phase giving rise to compactification. In this model, the eternal inflation-compactification is a problem whose solution is based on the tunneling of the negative cosmological constant to zero in the context of a dilaton field model having a potential with two local minima. The quantum inflation-deflation (QID) era is then realized by oscillations of the dilaton field around the absolute minimum (where $\Lambda < 0$). After tunneling to $\Lambda \simeq 0$ this quantum phase disappears allowing a classical description at later times.

There are differences and similarities between the models described above and the one presented in this paper. In our model, the metric is a 5-dimensional Kaluza-Klein with a negative cosmological constant whereas in [9], the metric is 10-dimensional with a cosmological constant which is positive. However, both models use signature transition as the process for addressing compactification, but through two completely different mechanisms. As for the

model presented in [8], we find a similarity in their assumption of a negative cosmological constant, but their compactification mechanism and matter contents are very different. They use either dust or a dilaton field as the matter content, contrary to our model in which there is no matter. However, in spite of the above differences, the behaviour of the two scale factors R and a in the present model merits some discussion under various possible choices of Λ in order to compare them with that of [8, 9] at the formal level. To this end, we first discuss signature transition in the model presented here by seeking suitably smooth continuous solutions for R and a passing through the hypersurface of signature change $\beta = 0$.

The classical solutions (14, 15) describe an empty Kaluza-Klein universe with a *negative* cosmological constant. When $\epsilon = -1$, the universe takes the same constant scale factor R_c in both Euclidean and Lorentzian regions, hence it is continuous at $\beta = 0$. The β dependent scale factor $a(\beta)$ is unbounded in the Euclidean region $\beta < 0$, passing continuously through $\beta = 0$ and exhibiting bounded oscillations in the Lorentzian region $\beta > 0$. The reality conditions on R_c and $a(\beta)$ forces k to be negative, thus rendering this universe open with $k = -1$ ². For $|\Lambda| \gg 0$ the solution (14) will give rise to a small constant scale factor R passing continuously from Euclidean to Lorentzian regions. The scale factor a will be enormously large in both regions compared to the scale factor R and passes through $\beta = 0$ continuously. The scale factor R is now compactified to a small size of order $|\Lambda|^{-\frac{1}{2}}$. Taking $|\Lambda| \simeq 0$, it is seen that for $\beta \ll 0$, $a(\beta)$ can become large and for $\beta \leq 0$, it tends to become very small in comparison with R_c . It passes through $\beta = 0$ continuously with a very small value $a(0) = R_c^{-1}$ and oscillates for $\beta > 0$ with amplitude $a(0)$ and a varying frequency. One of the most interesting features in this case is that signature transition now induces compactification on the scale factor a in the Lorentzian region, dragging it to a small size of order $|\Lambda|^{\frac{1}{2}}$. The first zero of this oscillatory function in the Lorentzian region occurs at

$$\beta_0 = \left(\frac{3\pi}{4} \sqrt{\frac{3}{2|\Lambda|}} \right)^{2/3}.$$

It is seen that the smallness of the cosmological constant (large β_0) allows for an extended Lorentzian region $0 \leq \beta < \beta_0$ which would correspond to a Kaluza-Klein cosmology with a large scale factor R and a stable compactified scale factor a , see figure 1. The long-time stability of the compactification is verified for the present bound on the cosmological constant $|\Lambda| \sim 10^{-56} \text{ cm}^{-2}$ since then $\beta_0 \geq \text{present age of universe}$.

When $\epsilon = +1$, the solutions (15) describe an empty Kaluza-Klein universe for which the internal space has zero size. Assuming a negative cosmological constant $\Lambda = 3k$ with $k = -1$, the solution $R(\beta)$ is real and behaves exponentially for $\beta < 0$, passing through $\beta = 0$ continuously and oscillating for $\beta > 0$. This, indeed, compactifies R to a small size $0 \leq R \leq \sqrt{2}$ in the Lorentzian region. Taking $|\Lambda| \simeq 0$ leads to a real scale factor $R(\beta)$. It has large values at both regions, behaving exponentially for $\beta \ll 0$, tending to a large constant value $\sim |\Lambda|^{-1/2}$ for $\beta \leq 0$, passing through $\beta = 0$ continuously and oscillating about this large value for $\beta > 0$. There is good agreement between R and its present observational bound $R \sim 10^{28} \text{ cm}$ for the choice $|\Lambda| \sim 10^{-56} \text{ cm}^{-2}$.

²The case $k = 0$ would give rise to a divergent a and zero R .

In conclusion, from the discussion given above, we find that choosing a small cosmological constant would lead to a good agreement between the size of the universe arising from the solutions (14, 15) and its present observational bound. This choice of Λ also leads to an agreement between the present unobservability of the size of the compactified dimension a and that resulting from the above solutions. As our solutions (14) show, we may recognize two hierarchial phases in the Lorentzian region as

$$t \simeq 0, |\Lambda| \gg 0 \Rightarrow R \sim |\Lambda|^{-1/2}, a \sim |\Lambda|^{1/2}$$

and

$$0 < t < t_0, |\Lambda| \simeq 0 \Rightarrow R \sim |\Lambda|^{-1/2}, a \sim |\Lambda|^{1/2}$$

both exhibiting the same relation $Ra \sim 1$ (t_0 being the present age of universe). Thus, we have found a classical link between the size of the extra dimension a and that of the visible dimension R in the Lorentzian region. This relation may account for the existence of a transformation $\Lambda \rightarrow \Lambda^{-1}$ leading to duality transformations $R \rightarrow a \sim R^{-1}$ and $a \rightarrow R \sim a^{-1}$. Therefore, such dualities may be studied in the context of a theory in which a large $|\Lambda|$ in the very early universe will result in a small $|\Lambda|$ at later times and hence an initial small R becomes very large while simultaneously the initial large a compactifies to a very small size. Therefore, if we regard the relation $Ra \sim 1$ as the main characteristic of this duality theory then switching off the large cosmological constant would be a natural result in order to be consistent with present observations.

A clear similarity is seen between our model and that of [8] in the sector of internal space compactification. Both of these models predict a deflationary phase for compactification of the internal dimensions before the onset of classical evolution. Considering equation (14) with $|\Lambda| \simeq 0$ and as is seen in figure 1, there is an exponentially decreasing $a(\beta)$ in the Euclidean region $-\infty < \beta \leq 0$ after which it undergoes a long-time stability in the Lorentzian region $0 \leq \beta < \beta_0$. Usually, the classical evolution of the universe begins at $\beta = 0$ and is considered to be in the Lorentzian sector so that the Euclidean sector may be assumed to be a pre-classical era [2]. Therefore, in the context of the present signature changing approach to compactification of the internal space a we also find a pre-classical era (Euclidean region) where a deflationary phase occurs after which the classical evolution begins. One may also resort to a mechanism in a more fundamental theory which would lead to a rapid deflationary phase (a large Λ) and then a long time stable phase (a small Λ) for $a(\beta)$. In fact, some switching off mechanisms could be proposed by which Λ could relax to zero.

An alternative similarity to [8] will be obtained if we consider our model as an effective part of a more fundamental theory in which a negative cosmological constant $\Lambda \sim 3k$ can tunnel to a very small value. Then, considering the Lorentzian solutions (14) with $\beta > 0$, and assuming $\Lambda \sim 3k$ we have two equally compact sizes $a \sim R$ initially at $t \simeq 0$. After the tunneling of $|\Lambda|$, the scale factor R becomes very large and a gets very small, both evolving classically thereafter. This picture may be identified, at least at the formal level, with the (QID) model in which, at first, all dimensions have equally compact sizes but after the (QID) phase ends, some of them become larger and the remaining ones get smaller. The cosmological constant can then tunnel from a large negative value to zero and the universe can be described classically afterwards.

Comparison with [9] shows that in our mini-superspace (u, v) , one finds, as in [9], two branches $u = \pm v$ corresponding to the vanishing of potential $W = \frac{1}{2}\omega^2(u^2 - v^2) = 0$, c.f. equation (7). The branch $u = -v$ of the curve $W = 0$ gives the compactification either of R or a according to the choice of the heirarchical cosmological constants. The other branch $u = +v$ leads to the compactification of a to zero and R to $0 \leq R \leq \sqrt{2}$.

As before, of particular interest is the branch $u = -v$ with a very small negative (magnitude) cosmological constant in which the size of compactification of the internal scale factor a , in the Lorentzian region, is of order $|\Lambda|^{\frac{1}{2}}$. At first glance, this result seems to be in conflict with that of [9] where $a \sim \Lambda^{-\frac{1}{2}}$ (for positive Λ), but since the cosmological constant in [9] is assumed to be large and the one here is small, we see that the results are in agreement ³. The only difference between the result of the compactification mechanisms in [9] and that proposed here concerns the scale factor R . In [9], the size of S^3 undergoes an eternal inflation whereas the scale factor R is constant here and in agreement with its observational bound.

5 Quantum Cosmology

One of the most interesting topics in the context of quantum cosmology is the mechanisms through which classical cosmology may emerge from quantum theory. When does a Wheeler-DeWitt wave function predict a classical space-time? Indeed, any attempt in constructing a viable quantum gravity requires understanding the connections between classical and quantum physics. Much work has been done in this direction over the past decade. Actually, there is some tendency towards using semiclassical approximations in dividing the behaviour of the wave function into two types, oscillatory or exponential which are supposed to correspond to classically allowed or forbidden regions. Hartle [11] has put forward a simple rule for applying quantum mechanics to a single system (universe): If the wave function is sufficiently peaked about some region in the configuration space we predict to observe a correlation between the observables which characterize this region. Halliwell [12] has shown that the oscillatory semiclassical WKB wave function is peaked about a region of the mini-superspace in which the correlation between the coordinate and momentum holds good and stresses that both *correlation* and *decoherence* are necessary before one can say a system is classical. Using Wigner functions, Habib and Laflamme [13] have studied the mutual compatibility of these requirements and shown that some form of coarse graining is necessary for classical prediction from WKB wave functions. Alternatively, Gaussian or coherent states with sharply peaked wave functions are often used to obtain classical limits by constructing wave packets.

A new aspect arises in quantum cosmology if the wave packets are to be constructed which would trace out a classical trajectory. In this case, the normalizability of the wave function is needed in order to have a correlation between classical and quantum cosmology. Of course, because of the superposition principle, some interference between the coherent states exist but enlarging the configuration space by adding a large number of higher degrees of freedom interacting with the mini-superspace variables leads to a decoherence effect.

³To see this, one may compare the long-time behaviour and sizes of a in the Lorentzian sector of figure 1 here, and figure 2 in [9].

Recently, this aspect of correspondence between classical and quantum cosmology has become of interest [14], especially in the context of signature transition [5, 6]. In [5], the authors have exactly solved the Wheeler-DeWitt equation in the form of an anisotropic oscillator-ghost-oscillator constraint and constructed states that highlight the classical trajectories and admit signature transition without resort to WKB approximations, hence avoiding the decoherence problem. Also in [15], it is shown that a large subset of generalized two-dimensional dilaton gravity models are dynamically equivalent to the isotropic oscillator-ghost-oscillator constraint and there may be correspondence between classical and quantum configurations like that obtained in [6].

In this section we shall concentrate on those aspect of correspondence between classical and quantum cosmology which are studied in [5]. The Lagrangian in (7) describes a classical Kaluza-Klein cosmology. The corresponding quantum cosmology is described by the Wheeler-DeWitt equation resulting from Hamiltonian (10) and can be written as

$$\left[\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - (u^2 - v^2)\omega^2 \right] \Psi(u, v) = 0 \quad (16)$$

where $\omega^2 \equiv -\frac{2\Lambda}{3}$. This is an isotropic oscillator-ghost-oscillator constraint. For $\Lambda < 0$, the “zero energy” solutions belong to a subspace of the Hilbert space spanned by separable eigenfunctions of a 2-dimensional isotropic simple harmonic oscillator Hamiltonian, and can be written as

$$\Phi_{(n_1, n_2)}(u, v) = \alpha_{n_1}(u)\beta_{n_2}(v) \quad n_1, n_2 = 0, 1, 2, \dots \quad (17)$$

with

$$\alpha_n(u) \equiv \left(\frac{\omega}{\pi}\right)^{1/4} \frac{e^{-\frac{\omega u^2}{2}}}{\sqrt{2^n n!}} H_n(\sqrt{\omega}u), \quad (18)$$

$$\beta_n(v) \equiv \left(\frac{\omega}{\pi}\right)^{1/4} \frac{e^{-\frac{\omega v^2}{2}}}{\sqrt{2^n n!}} H_n(\sqrt{\omega}v) \quad (19)$$

provided $(n_1 + \frac{1}{2})\omega = (n_2 + \frac{1}{2})\omega$ with the restriction $n_1 = n_2 \equiv n$ ⁴. $H_n(x)$ is the Hermit polynomial and the eigenfunctions are normalized according to

$$(\alpha_n, \alpha_m) = \delta_{n,m}, \quad (\beta_n, \beta_m) = \delta_{n,m} \quad (20)$$

where $(\ , \)$ denotes the inner product. The solutions $\Phi_{(n,n)}(u, v)$ span a Hilbert subspace of measurable square integrable functions on R^2 in the form

$$\Psi(u, v) = \sum_{n=0}^{\infty} c_n \Phi_{(n,n)}(u, v) \quad (21)$$

where $c_n \in C$ and

$$(\Phi_{(n,n)}, \Phi_{(n',n')}) = \delta_{n,n'}. \quad (22)$$

We are interested in constructing a coherent wave packet with good asymptotic behaviour in the mini-superspace, peaking in the vicinity of one of the classical loci $u = \pm v$ in the

⁴Note that in [5] there is a quantization condition on ω resulting in distinct Hilbert subspaces corresponding to excited cosmologies.

$\{u, v\}$ configuration space. It is well-known that for a harmonic oscillator, non-dispersive wave packets may be constructed by superposition of energy eigenfunctions. Therefore, the wave packet (21) is what we need. We take the solution as being represented by equation (21) with the sum to be truncated at a suitable value of n displaying this peak. Figures 2 and 3 show surface and density plots of $|\Psi(u, v)|^2$, where we have taken the combinations from 6 terms with all c_n up to c_5 taken to be unity. Taking more terms would only have a small effect on the results. It is seen that a good correlation exists between these patterns and the classical loci $u = v$ in the configuration space $\{u, v\}$.

Wave packets in the mini-superspace can only be understood as unparametrized tubes. This is because the Wheeler-DeWitt equation does not have an intrinsic time parameter. However, in order to relate the properties of the wave packet (21) to an evolving classical cosmology with classical time in the Lorentzian region, one may take $\{\beta\}$ as a family of coordinate functions labeling some subset of the real line containing the point $\beta = 0$. Then the loci $u = \pm v$ admit parametrizations in terms of $\{\beta\}$

$$u(\beta) = A \cosh \left(\sqrt{\frac{2\Lambda}{3}} \frac{2}{3} \beta^{\frac{3}{2}} \right), \quad v(\beta) = \epsilon A \cosh \left(\sqrt{\frac{2\Lambda}{3}} \frac{2}{3} \beta^{\frac{3}{2}} \right) \quad (23)$$

with the point $\beta = 0$ implying a transition from Euclidean to Lorentzian signature. Now, a change of coordinate $\beta \rightarrow \beta' = F(\beta)$ will induce a change of parametrization for the classical loci and, for $\beta > 0$ with F monotonic, would correspond to an alternative choice of classical time. To the extent that the classical loci are highlighted by the state (21) we say that the family of classical times, regarded as alternative parametrization of such loci, arise dynamically from this state.

6 Conclusions

In this paper we have considered the solutions of Einstein equations in an empty (4+1) dimensional Kaluza-Klein cosmology with a Robertson-Walker type metric having a negative cosmological constant. These solutions admit a degenerate metric signifying a signature transition from Euclidean to Lorentzian domains. Motivated by the subject of compactification which has been studied in the context of signature change for a positive cosmological constant, we have shown that here, signature transition can provide a compactification either for the usual scale factor R or the internal scale factor a according to the choice of hierarchical negative cosmological constant. The most interesting and desirable case emerges by taking a very small negative cosmological constant which would then predict the relation $Ra \sim 1$ and compactification of a as $a \sim |\Lambda|^{1/2}$ for the Lorentzian space-time and an exponentially damping behaviour (deflation) of a for the Euclidean region. These results are formally in general agreement with those obtained in [9] and [8]. The corresponding Wheeler-DeWitt equation is exactly solved to construct a state that may be identified with a non-dispersive wave packet peaking in the vicinity of the classical Kaluza-Klein submanifold admitting signature transition. This remarkable correspondence would help us to look at the existence of higher dimensional geometries undergoing a continuous change of signature as semi-classical limits in the context of quantum cosmology.

We may remark that the model presented here does not predict the standard big-bang but rather an eternal cosmology. One way of avoiding this eternal universe would be to assume an adiabatically evolving Λ . Alternatively, one may work within the context of a dilaton field model in which a typical dilaton field ϕ with its potential $V(\phi)$ results in an effective cosmological constant Λ . Thus, if the dilaton model admits either a duality transformation $|\Lambda| \rightarrow |\Lambda|^{-1}$ or tunneling from $|\Lambda|$ to zero leading to switching off $|\Lambda|$, then it will provide for the evolution of the scale factor R from small to large scales. Nevertheless, in its present form this eternal universe with no big-bang and a size which is large enough (adjusting $|\Lambda|$ to a very small constant) may have the advantage of being compatible with the present observations and seems to be free of the flatness problem.

It is also worth noting that if one takes a positive cosmological constant rather than negative, one ends up with having an inflationary rather than a compactification phase in the Lorentzian solutions. Finally, we remark that for $(4+D)$ dimensional Kaluza-Klein cosmology with $D > 1$ in the present model, the quantization is a difficult problem which requires further investigations.

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Figure Captions

Figure 1. Variation of $a(\beta)$ with β for a typical small value of $\Lambda \sim 10^{-3}$.

Figure 2. Surface plot of $|\Psi(u, v)|^2$.

Figure 3. Density plot of $|\Psi(u, v)|^2$.

Figure 1.

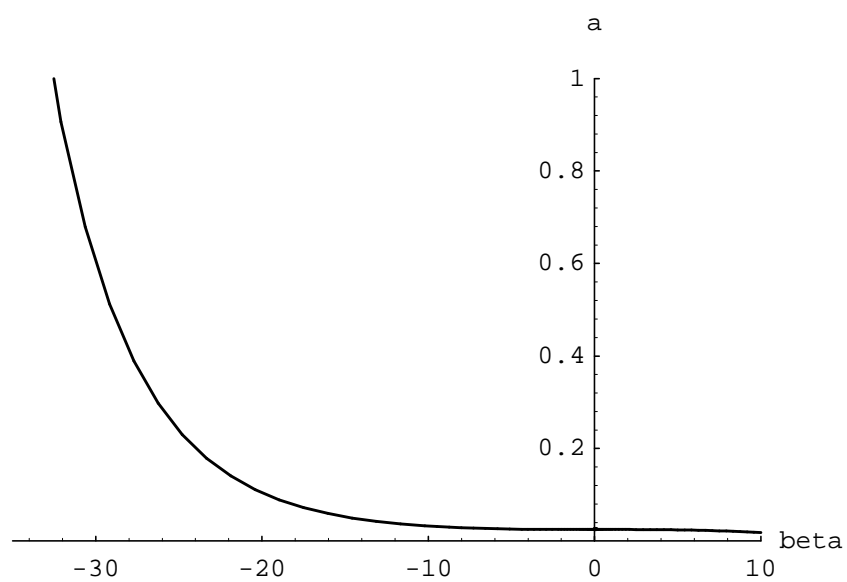


Figure 2.

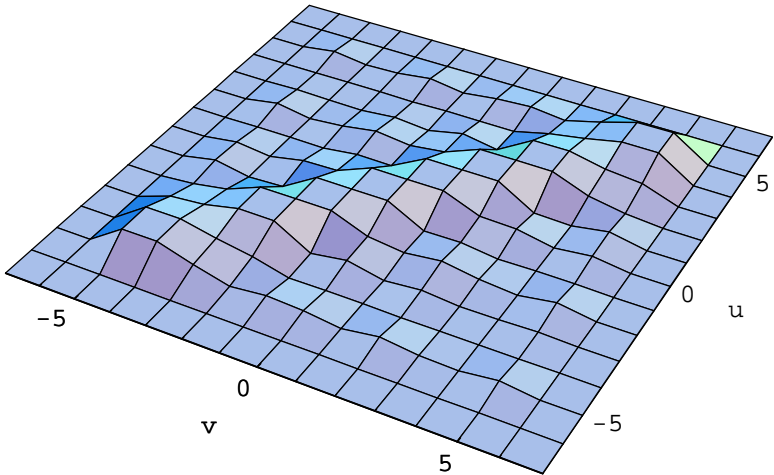


Figure 3.

